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ESTIMATES ON $\mu(z)$-HOMEOMORPHISMS OF THE UNIT DISK

BY

CHEN ZHIGUO

*Department of Mathematics, XiXi Campus, Zhejiang University
Hangzhou, Zhejiang 310028, P. R. China
e-mail: zgchen1205@163.net*

ABSTRACT

In the theory of K-quasiconformal mappings, Mori's theorem shows that K-quasiconformal mappings on the unit disk satisfy the Hölder condition, where the coefficient 16 is best possible. In this paper, we prove that self-$\mu(z)$-homeomorphisms on the unit disk have an analogical result to Mori's theorem when the integral mean dilatations are controlled by log function. An unimprovable inequality is obtained.

1. Introduction

Let us recall the analytic definition of K-quasiconformal mappings.

Definition: Let f be a sense-preserving homeomorphism in the plane. We call f a K-quasiconformal mapping if it satisfies the following two conditions:

(i) f is absolutely continuous on lines in Ω.

(ii) $|f_{\bar{z}}| \leq k|f_z|$, for almost all $z \in \Omega$.

It is well known that a K-quasiconformal mapping f with two points normalized ($f(a_1) = b_1, f(a_2) = b_2$, where $a_1 \neq a_2, b_1 \neq b_2$) satisfies a uniform Hölder condition on any compact subset:

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2|^{1/K}. \tag{1.1}$$

Particularly, when f is a K-quasiconformal mapping of the unit disk Δ onto itself with normalization $f(0) = 0$, the following famous Mori's result holds:

$$|f(z_1) - f(z_2)| < 16|z_1 - z_2|^{1/K}, \tag{1.2}$$

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where the coefficient 16 is best possible.

In recent years, many mathematicians have been interested in a more generalized class of locally quasiconforml mappings. In this paper, we examine $\mu(z)$-homeomorphisms.

Definition: Let E be a compact set in Ω which is of σ-linear measure. Let $\mu(z)$ be a measurable function in Ω. Function f is called a $\mu(z)$-homeomorphism, if f is a sense-preserving homeomorphism in Ω and is locally $\mu(z)$-quasiconformal in $\Omega - E$.

The importance of $\mu(z)$-homeomorphisms lies in the fact that, on one hand, they are natural generalizations of K-quasiconformal mappings; on the other hand, all of them come from the study of homeomorphic solutions of the following Betrami equation,

$$f_{\bar{z}} = \mu(z) f_z. \tag{1.3}$$

We refer readers on $\mu(z)$-homeomorphisms to [2], [3], [4], [5], [6], [8], [9], [10], [11] and [12]. It should be pointed out that until now, the theory of $\mu(z)$-homeomorphisms is far from complete.

Throughout this paper, we restrict ourself to the unit disk Δ. Denote the dilatation function of f by $D_f(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$. Let $\gamma(r)$ be an circular arc in Δ with radius r. Denote the integral mean dilatation on $\gamma(r)$ by

$$D_f^*(\gamma(r)) = \frac{1}{L(\gamma(r))} \int_{\gamma(r)} D_f(z)|dz|, \tag{1.4}$$

where $L(\gamma(r))$ represents the linear length of $\gamma(r)$. In particular, if $\gamma(r) = \{z | z = z_0 + re^{i\theta},\ 0 \le \theta < 2\pi\}$, then

$$D^*(\gamma(r)) = D^*(z_0, r) = \frac{1}{2\pi} \int_0^{2\pi} D(z_0 + re^{i\theta}) d\theta.$$

The intention of this paper is to estimate $|f(z_1) - f(z_2)|$ under self-$\mu(z)$-homeomorphisms of Δ. We control the growth order of $D^*(\gamma(r))$ by a log function and obtain the following main result:

THEOREM 1.1: *Suppose that f is a self-$\mu(z)$-homeomorphism of Δ with normalization $f(0) = 0$. If for any $r < \frac{1}{2}$, $\gamma(r) \subset \Delta$, there exists a constant M such that*

$$D^*(\gamma(r)) \le M \log(e/r), \tag{1.5}$$

then

$$\lim_{|z_1-z_2|\to 0}\sup_{z_1,z_2\in\Delta}\frac{\log|f(z_1)-f(z_2)|}{\log\log\frac{2e}{|z_1-z_2|}}\le -1/M \tag{1.6}$$

where the constant $-1/M$ is best possible.

2. Some lemmas

To prove the theorem, we shall prepare a few lemmas in this section.

Let Γ be a family of curves in the plane. Each $\gamma\in\Gamma$ shall be a countable union of open arcs, closed arcs or closed curves, and every closed subarc shall be rectifiable. We use the properties of the extremal length of Γ. The definition is as follows.

1. ρ is non-negative Borel function.
2. $A(\rho)=\int\int_{\mathbb{C}}\rho^2dxdy\neq 0,\infty$.

For such a ρ, set

$$L_\gamma(\rho)=\int_\gamma \rho|dz|$$

if ρ is measurable on γ. Otherwise, $L_\gamma(\rho)=\infty$. We introduce

$$L(\rho)=\inf_{\gamma\in\Gamma}L_\gamma(\rho)\quad\text{and}\quad \lambda(\Gamma)=\sup_\rho\frac{L(\rho)^2}{A(\rho)}$$

for all allowable ρ.

Let $f_j(x)$ be continuous on line segment $[a_1,a_2]$ $(j=1,2)$, and $f_1(x)<f_2(x)$ for any $x\in[a_1,a_2]$. Denote

$$Q=\{(x,y)|a_1<x<a_2,\ f_1(x)<y<f_2(x)\}.$$

Let $f(z)$ be a $\mu(z)$-homeomorphism of Q. Define the module of Q by the extremal length of Γ, a family of curves which connect $\{z|x=a_1\}$ with $\{z|x=a_2\}$ in Q, namely, $\mathrm{mod}(Q)=\lambda(\Gamma)$. Correspondingly, $\mathrm{mod}\,f(Q)$ is defined as the extremal length of $f(\Gamma)$. We estimate the module of $f(Q)$ as follows.

LEMMA 2.1: *If the dilatation function $D(z)$ of f is integrable in Q, then*

$$\int_{a_1}^{a_2}\frac{dx}{\int_{f_1(x)}^{f_2(x)}D(x+iy)dy}\le \mathrm{mod}\,f(Q). \tag{2.1}$$

Proof: There exists a conformal mapping ϕ which maps $f(Q)$ onto a rectangle $R=\{(u,v)|0\le u\le v, 0\le v\le b\}$. Set $h=\phi\circ f$. From the conformal invariance of extremal length, we have

$$\mathrm{mod}\,f(Q)=\mathrm{mod}\,h(Q)=a/b. \tag{2.2}$$

Now, we prove that h is absolutely continuous on lines. In fact,

$$\iint_Q (|h_z| + |h_{\bar z}|)dxdy = \iint_Q D_h(z)^{\frac12} J_h(z)^{\frac12} dxdy$$
$$= \iint_Q D_f(z)^{\frac12} J_h(z)^{\frac12} dxdy,$$

where $J_h(z)$ is the Jacobian of h, namely $J_h = |h_z|^2 - |h_{\bar z}|^2$.

Therefore, applying the Schwarz inequality to the right side of the last equality, we have

$$\iint_Q (|h_z| + |h_{\bar z}|)dxdy < \left(\iint_Q D_f(z)dxdy\right)^{\frac12} \left(\iint_Q J_h(z)dxdy\right)^{\frac12}.$$

From the integrability of $D_f(z)$ and $J_h(z)$, $|h_z|$ and $|h_{\bar z}|$ are integrable. Thus, h is ACL (see lemma 2 in chapter II of [1]).

From the fact that $h(x+iy)$ is absolutely continuous for almost all $x \in [a_1, a_2]$, it follows that

$$b \le \int_{f_1(x)}^{f_2(x)} |h_y(x+iy)|dy. \tag{2.3}$$

Set

$$g(z) = |h_y(z)|^2 / J_h(z).$$

By the chain rule one can show that

$$g(z) = \frac{|1-\mu_h(z)|^2}{1-|\mu_h(z)|^2} = \frac{|1-\mu_f(z)|^2}{1-|\mu_f(z)|^2} \le D_f(z). \tag{2.4}$$

The above inequality together with the hypothesis of the lemma yield that $g(z)$ is integrable in Q. Set

$$g_n(z) = \begin{cases} g(z), & |g(z)| \le n; \\ 0, & |g(z)| > n. \end{cases}$$

Denote

$$I_n(x) = \int_{f_1(x)}^{f_2(x)} g_n(z)dy.$$

By Fubini's theorem, $I_n(x)$ exists for almost all $x \in [a_1, a_2]$ and a given n. Obviously for a fixed x, the sequence $\{I_n(x)\}$ is increasing with n. So the sequence $I_n(x)$ converges if the limit is permitted to be $+\infty$. This means that the integral

$\int_{f_1(x)}^{f_2(x)} g(z)dy$ exists for almost all $x \in [a_1, a_2]$. The same argument is available for $J_h(z)$. Now it is not difficult to prove that

$$b^2 / \int_{f_1(x)}^{f_2(x)} g(z)dy \leq \int_{f_1(x)}^{f_2(x)} J_h(z)dy. \tag{2.5}$$

Because the above inequality holds if

$$\int_{f_1(x)}^{f_2(x)} g(z)dy = +\infty \quad \text{or} \quad \int_{f_1(x)}^{f_2(x)} J_h(z)dy = +\infty,$$

we may assume that both of them are finite. Applying Schwarz inequality to (2.3), we have

$$b^2 \leq \left(\int_{f_1(x)}^{f_2(x)} \sqrt{g(z)}\sqrt{J_h(z)}dy \right)^2 \leq \int_{f_1(x)}^{f_2(x)} g(z)dy \int_{f_1(x)}^{f_2(x)} J_h(z)dy. \tag{2.6}$$

Thus (2.5) immediately follows from (2.6).

Integrating both sides of inequality (2.5) over $x \in [a_1, a_2]$, we get

$$\int_{a_1}^{a_2} \frac{b^2 dx}{\int_{f_1(x)}^{f_2(x)} g(z)dy} \leq \int_{a_1}^{a_2} \int_{f_1(x)}^{f_2(x)} J_h(z)dydx \leq \text{mes}(h(Q)) = ab.$$

Hence, from (2.2), (2.4) and the above inequality, (2.1) follows. This completes the proof of the lemma. ∎

Let $\theta_j(r)$ be continuous on $[r_1, r_2]$, and $0 < \theta_2(r) < \theta_1(r) < 2\pi$. Denote

$$\begin{aligned} P =& P(z_0; r_1, r_2; \theta_1(r), \theta_2(r)) \\ =& \{z | z = z_0 + re^{i\theta}, r_1 < r < r_2, \theta_1(r) < \theta < \theta_2(r)\}. \end{aligned}$$

Function $\log(z - z_0)/r_1$ conformally maps P onto Q. From Lemma 2.1 and the conformal invariance of module, there follows

LEMMA 2.2: *Suppose that $f(z)$ is a $\mu(z)$-homeomorphism in P. If $D_f(z)$ is integrable in P, then*

$$\int_{r_1}^{r_2} \frac{dr}{r \int_{\theta_1}^{\theta_2} D_f(z_0 + re^{i\theta})d\theta} \leq \text{mod}\, f(P) \tag{2.7}$$

where $\text{mod}\, f(P)$ is defined as the extremal length of $f(\Gamma)$ and Γ is a family of curves which connect the corresponding two arcs in P.

Particularly, if P is an annulus, we have

$$\int_{r_1}^{r_2} \frac{dr}{2\pi r D_f^*(z_0, r)} \leq \text{mod}\, f(P). \tag{2.8}$$

For the proof of the above inequality when f is a K-quasiconformal mapping, we refer to [7] and [11].

Set

$$A_\Delta(z_0; r_1, r_2) = \{z | r_1 < |z - z_0| < r_2, z_0 \in \Delta\} \cap \Delta.$$

By (2.7), the module of $f(A_\Delta(z_0; r_1, r_2))$ has the following estimate:

$$\int_{r_1}^{r_2} \frac{dr}{2\pi D^*(\gamma(r))} \leq \operatorname{mod} f(A_\Delta(z_0; r_1, r_2)) \tag{2.9}$$

where $D^*(\gamma(r))$ is the integral mean dilatation on $\gamma(r)$.

LEMMA 2.3: *Suppose that $f(z)$ is a $\mu(z)$-homeomorphism in the unit disk Δ. If $D_f^*(\gamma(r))$ satisfies (1.5), then the image $f(\Delta)$ is conformally equivalent to the unit disk Δ and f can be topologically extended to the boundary $\partial\Delta$.*

Proof: From the topological point of view, homeomorphism f maps Δ either onto the whole plane or onto a simply-connected domain of hyperbolic type.

Suppose that f maps Δ onto the whole plane. We consider domain $A_\Delta(z_0; \delta, 2\delta)$ where $z_0 \in \partial\Delta$, $\delta < \frac{1}{4}$. By the definition of module and extremal length, $\operatorname{mod} f(A_\Delta(z_0; \delta, 2\delta)) = 0$. On the other hand, from the hypothesis of the lemma, we deduce that $D_f(z)$ is integrable in $A_\Delta(z_0; \delta, 2\delta)$. Hence, $\mu(z)$-homeomorphism f satisfies conditions of Lemma 2.2. Using (2.9) and (1.5), we have

$$0 < \int_{\delta}^{2\delta} \frac{dr}{2\pi M r \log(e/r)} \leq \operatorname{mod} f(A_\Delta(z_0; \delta, 2\delta)). \tag{2.10}$$

This is a contradiction. Thus $f(\Delta)$ must be conformally equivalent to Δ.

From the Riemann mapping theorem, there exists a conformal mapping ϕ such that $\phi \circ f(\Delta) = \Delta$. By definition in §1, $D_f^*(\gamma(r)) = D_{\phi\circ f}^*(\gamma(r))$. So, without loss of generality, we may assume that $f(\Delta) = \Delta$.

Next, in three steps we prove that f can be homeomorphically extended to the boundary.

Firstly, we prove that for any $z_0 \in \partial\Delta$, $f(z)$ converges as z tends to z_0 in Δ.

Otherwise, there exist two sequences $\{z_n^j\}$ $(j = 1, 2)$ tending to z_0 such that $f(z_n^j) \to u_j$, where u_1 and u_2 are two different points on $\partial\Delta$. Consider domain $A_\Delta(z_0; r, \frac{1}{2})$. Let $\operatorname{mod} f(A_\Delta(z_0; r, \frac{1}{2})) = \lambda(f(\Gamma_r))$, where Γ_r is a family of curves which connect $\{z||z - z_0| = \frac{1}{2}\}$ with $\{z||z - z_0| = r\}$ in $A_\Delta(z_0; r, \frac{1}{2})$. By (2.9) and conditions of the lemma, $\operatorname{mod} f(A_\Delta(z_0; r, \frac{1}{2})) \to +\infty$ as $r \to 0$. Consequently,

$$\lambda(f(\Gamma_r)) \to +\infty, \quad \text{as } r \to 0. \tag{2.11}$$

Set $U_\Delta(z_0;\frac{1}{2}) = \{z||z-z_0| < \frac{1}{2}\}\cap\Delta$. The curve $f(|z-z_0| = \frac{1}{2})$ intersects $\partial\Delta$ at v_1 and v_2. Let $Q(v_1, v_2, u_1, u_2)$ be a quadrilateral with domain $f(U_\Delta(z_0,\frac{1}{2}))$ and vertices v_1, v_2, u_1, u_2. The module $\operatorname{mod} Q(v_1, v_2, u_1, u_2)$ is defined as the extremal length $\lambda(\Gamma)$, where Γ is a family of curves which connect the curve $f(|z-z_0| = \frac{1}{2})$ and arc $\widehat{u_1u_2}$. It is easy to see that $\lambda(\Gamma) < \infty$. Since $f(\Gamma_r) < \Gamma$, by monotonicity of extremal length, $\lambda(f(\Gamma_r)) < \lambda(\Gamma) < \infty$, which contradicts (2.11). This implies that $f(z)$ converges.

Secondly, from the above discussion, it makes sense that $f(z_0) = \lim_{z\to z_0} f(z)$ whenever z tends to z_0 in Δ. Analogous reasoning yields the continuity of $f|_{\partial\Delta}$.

Finally, we shall prove that $f|_{\partial\Delta}$ is injective.

Otherwise, there exist w and $z_1, z_2 \in \partial\Delta$ such that $f(z_1) = f(z_2) = w$. From topology, f maps arc $\widehat{z_1z_2}$ to w. Set $z_0 = (z_1+z_2)/2$, $r_1 = |z_2 - z_1|/4$ and $r_2 = |z_2 - z_1|/2$. By definition of module, $\operatorname{mod} f(A_\Delta(z_0; r_1, r_2)) = 0$. However, from conditions of the lemma and (2.8), it follows that $\operatorname{mod} f(A_\Delta(z_0; r_1, r_2)) > 0$. This contradiction shows that $f|_{\partial\Delta}$ is injective. The proof is complete. ∎

Remark: Professor Li has obtained a homeomorphic extension of locally quasiconformal mappings under conditions different from this paper (see [10]).

3. Proof of the main result

Let $G(R)$ be a symmetric Grötzschian domain and $M(\lambda)$ be a symmetric Mori's domain. Their moduli are represented by functions $\frac{1}{2\pi}\log\Phi(R)$ and $\frac{1}{2\pi}\log\chi(\lambda)$, respectively. In this section we need following two inequalities:

$$\Phi(R) \le 4(R), \tag{3.1}$$

$$\lambda\chi(\lambda) \le 16. \tag{3.2}$$

For more details about functions Φ and χ, we refer to [1].

Proof of Theorem 1.1: In view of the hypothesis of the theorem and Lemma 2.3, f can be topologically extended to $\overline{\Delta}$. Extend f by

$$f\left(\frac{1}{\bar{z}}\right) = \frac{1}{\overline{f(z)}}.$$

Then

$$|\mu_f(z)| = |\mu_f(1/\bar{z})|. \tag{3.3}$$

Therefore, f is a $\mu(z)$-homeomorphism of the whole plane.

When f is K-quasiconformal, the above extended f is still K-quasiconformal in the whole plane. But for $\mu(z)$-homeomorphisms satisfying (1.5), generally, the constant M will change after extension. We claim that

$$D_f^*(z_0, r) \leq (1+r)^2 M \log(e/r) \tag{3.4}$$

holds for all $z_0 \in \overline{\Delta} \cap \{|z| > \frac{3}{4}\}$ when $r \leq \frac{1}{4}$.

Set $C_i(r) = \{z \mid |z - z_0| = r\} \cap \Delta$, which is a circular arc inside Δ. Set $C_o(r) = \{z \mid |z - z_0| = r\} \cap (C - \Delta)$, which is a circular arc outside Δ. Let $\overline{C_o(r)}$ be the image of $C_o(r)$ under reflection $1/\bar{z}$. Hence it is a circular arc inside the unit disk Δ. Let $\bar{r}$ be the radius of $\overline{C_o(r)}$. By direct calculation, one can show that

$$\bar{r} = \frac{r}{||z_0|^2 - r^2|}. \tag{3.5}$$

Since $r \leq \frac{1}{4}$ and $\frac{3}{4} \leq |z_0| \leq 1$, from (3.5) it follows that

$$r < \bar{r} < 2r < \frac{1}{2}. \tag{3.6}$$

By hypothesis of the theorem and (3.6), we have

$$\frac{\int_{\overline{C_o(r)}} D_f(z)|dz|}{\int_{\overline{C_o(r)}} |dz|} \leq M \log \frac{e}{\bar{r}} < M \log \frac{e}{r}. \tag{3.7}$$

The inequality (3.3) implies that $D_f(z) = D_f(1/\bar{z})$. Hence

$$\begin{aligned} \frac{\int_{\overline{C_o(r)}} D_f(z)|dz|}{\int_{\overline{C_o(r)}} |dz|} &= \frac{\int_{C_o(r)} D_f(1/\bar{z})|d\frac{1}{\bar{z}}|}{\int_{C_o(r)} |d\frac{1}{\bar{z}}|} \\ &= \frac{\int_{C_o(r)} D_f(z)|d\frac{1}{\bar{z}}|}{\int_{C_o(r)} |d\frac{1}{\bar{z}}|} = \frac{\int_{C_o(r)} \frac{D_f(z)}{|z|^2}|dz|}{\int_{C_o(r)} \frac{1}{|z|^2}|dz|} \\ &\geq \frac{\min_{z \in C_0(r)} |z|^2}{\max_{z \in C_0(r)} |z|^2} \frac{\int_{C_o(r)} D_f(z)|dz|}{\int_{C_o(r)} |dz|} \\ &= \frac{1}{(1+r)^2} \frac{\int_{C_o(r)} D_f(z)|dz|}{\int_{C_o(r)} |dz|}. \end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned} \frac{\int_{C_o(r)} D_f(z)|dz|}{\int_{C_o(r)} |dz|} &\leq (1+r)^2 \frac{\int_{\overline{C_o(r)}} D_f(z)|dz|}{\int_{\overline{C_o(r)}} |dz|} \\ &\leq (1+r)^2 M \log \frac{e}{r}. \end{aligned} \tag{3.9}$$

Again using the hypothesis of the theorem, we have

$$\frac{\int_{C_i(r)} D_f(z)|dz|}{\int_{C_i(r)} |dz|} \leq M \log \frac{e}{r}. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\begin{aligned} D_f^*(z_0, r) &= \frac{\int_{C_o(r)} D_f(z)|dz| + \int_{C_i(r)} D_f(z)|dz|}{\int_{C_o(r)} |dz| + \int_{C_i(r)} |dz|} \\ &\leq (1+r)^2 M \log(e/r). \end{aligned}$$

So the claim is proved.

Let $r_2 \leq \frac{1}{4}$. For any distinct points z_1, z_2 in Δ while $|z_1 - z_2| < r_2$, we construct an annulus

$$A = \{z \mid |z_1 - z_2|/2 < |z - (z_1 + z_2)/2| < r_2\}.$$

Considering two possibilities, we shall estimate on $|f(z_1) - f(z_2)|$ as follows.

CASE 1: A lies in the unit disk Δ. According to (2.7) and the hypothesis of the theorem,

$$\int_{r_1}^{r_2} \frac{dr}{2\pi M r \log \frac{e}{r}} \leq \int_{r_1}^{r_2} \frac{dr}{2\pi r D^*(z_0, r)} \leq \operatorname{mod} f(A) \tag{3.11}$$

where $r_2 \leq \frac{1}{4}$, $r_1 = |z_2 - z_1|/2$ and $z_0 = (z_1 + z_2)/2$.

Let $\zeta_1 = f(z_1), \zeta_2 = f(z_2)$. Considering the mapping

$$w = \frac{\zeta - \zeta_1}{1 - \bar{\zeta}_1 \zeta}$$

and noting the inequality (3.1), we have

$$\operatorname{mod} f(A) \leq \frac{1}{2\pi} \log \Phi\Big(\Big|\frac{1 - \bar{\zeta}_1 \zeta_2}{\zeta_2 - \zeta_1}\Big|\Big) \leq \frac{1}{2\pi} \log \frac{8}{|\zeta_2 - \zeta_1|}. \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$\int_{r_1}^{r_2} \frac{dr}{2\pi M r \log \frac{e}{r}} \leq \frac{1}{2\pi} \log \frac{8}{|f(z_2) - f(z_1)|}.$$

After rearrangement, we obtain

$$|f(z_2) - f(z_1)| \leq 8\Big(\log \frac{e}{r_2}\Big)^{1/M} \Big(\log \frac{2e}{|z_2 - z_1|}\Big)^{-1/M}. \tag{3.13}$$

CASE 2: A does not lie in Δ, hence it does not contain the origin O. It also implies that $|z_0| > 1 - r_2 \geq 3/4$. Therefore (3.4) holds. Using (2.7) and (3.4), we have

$$\frac{1}{(1+r_2)^2}\int_{r_1}^{r_2}\frac{dr}{2\pi M r\log(e/r)} \leq \int_{r_1}^{r_2}\frac{dr}{2\pi r D^*(z_0,r)} \leq \operatorname{mod} f(A). \tag{3.14}$$

Denote $B(z_0, r)$ as a disk with center at z_0 and radius r. Now the image of the inner continuum of A intersects $\{\zeta||\zeta| < 1\}$ in a set with diameter $\geq |\zeta_2 - \zeta_1|$, and the outer continuum of A contains the origin. Therefore, by the extremal property of Mori's domain and inequality (3.2), we have

$$\operatorname{mod} f(A) \leq \frac{1}{2\pi}\log\chi(|\zeta_1-\zeta_2|) \leq \frac{1}{2\pi}\log(16/|\zeta_2-\zeta_1|). \tag{3.15}$$

From (3.14) and (3.15), we have

$$|f(z_2) - f(z_1)| \leq 16\Big(\log\frac{e}{r_2}\Big)^{1/M}\Big(\log\frac{2e}{|z_2-z_1|}\Big)^{-(\frac{1}{1+r_2})^2\frac{1}{M}}. \tag{3.16}$$

Comparing (3.13) with (3.16), we see that (3.16) holds whenever $|z_2 - z_1| < r_2 \leq \frac{1}{4}$.

Now we choose points z_1 and z_2 to be sufficiently close such that $|z_2 - z_1| < 2e^{-3}$. Let

$$x = \log\frac{2e}{|z_2-z_1|}, \quad r_2 = \frac{1}{x}.$$

Then $x > 4$ and $r_2 < \frac{1}{4}$. Rewrite (3.16) as

$$|f(z_2) - f(z_1)| \leq 16(\log x + 1)^{\frac{1}{M}} x^{-(\frac{1}{1+1/x})^2\frac{1}{M}}. \tag{3.17}$$

It is not difficult to see that the inequalities

$$\log x < x^{1/\sqrt{\log x}} \quad \text{and} \quad \Big(\frac{1}{1+1/x}\Big)^2 > 1 - \frac{2}{x} \tag{3.18}$$

hold whenever $x > 4$. Substituting (3.18) into (3.17), we have

$$\begin{aligned}|f(z_2) - f(z_1)| &\leq 16(ex)^{\frac{1}{M\sqrt{\log ex}}} x^{-(1-\frac{2}{x})\frac{1}{M}} \\ &< 16e^{\frac{1}{M}} x^{-\frac{1}{M}\left(1-\frac{2}{x}-\frac{1}{\sqrt{\log ex}}\right)}.\end{aligned}$$

From the above inequality, we immediately obtain

$$\lim_{|z_2-z_1|\to 0}\ \sup_{z_1,z_2\in\Delta}\frac{\log|f(z_2)-f(z_1)|}{\log\log\frac{2e}{|z_2-z_1|}} \leq -\frac{1}{M}.$$

Finally, it remains to give an example to show that the constant $-1/M$ is best possible. Set

$$f(re^{i\theta}) = \left(\log\frac{e}{r}\right)^{-1/M} e^{i\theta}.$$

Direct computation shows that

$$D(re^{i\theta}) = M\log(e/r).$$

Hence f is a self-$\mu(z)$-homeomorphism of Δ, and the exceptional set E contains only one point, the origin O. Let $z_1(r) = r$ and $z_2(r) = -r$. The distance between $f(r)$ and $f(-r)$,

$$|f(r) - f(-r)| = 2\left(\log\frac{e}{r}\right)^{-1/M},$$

shows that the estimates given in Theorem 1.1 are best possible. ∎

Now we come to the normal property of a family of $\mu(z)$-homeomorphisms whose integral mean dilatation functions satisfy inequality (1.5). A family is said to be normal if every sequence of its elements contains a subsequence which is locally uniformly convergent. Obviously, if a family is equicontinuous, it must be normal.

THEOREM 3.1: *Let $F = \{f_n\}$ be a family of self-$\mu(z)$-homeomorphisms of Δ with dilatation functions $D_n(z)$ satisfying (1.5). Then F is normal.*

Proof: Taking $r_2 = \frac{1}{4}$ in inequality (3.16), we infer that the inequality

$$|f(z_2) - f(z_1)| \leq 16(1+\log 4)^{\frac{1}{M}}\left(\log\frac{2e}{|z_2 - z_1|}\right)^{-\frac{16}{25}\frac{1}{M}} \tag{3.19}$$

holds when $|z_2 - z_1| < \frac{1}{4}$. If $|z_2 - z_1| \geq \frac{1}{4}$, (3.19) is trivial. Therefore, (3.19) is valid for all $z_1, z_2 \in \Delta$. We see that the upper bound of $|f(z_2) - f(z_1)|$ is independent of f, which implies the family F is equicontinuous. Hence F is normal. ∎

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